Assignment 2

① 1°, $\because \alpha(s) = (\cos s, \sin s, 0)$

$\alpha'(s) = (-\sin s, \cos s, 0)$

$w(s) = \alpha'(s) + e_3 = (-\sin s, \cos s, 1)$

$\therefore X(s,v) = \alpha(s) + v \cdot w(s)$

$= (\cos s - v\sin s, \sin s + v\cos s, v)$

$\therefore (\cos s - v\sin s)^2 + (\sin s + v\cos s)^2 - v^2$

$= 1 + v^2 - v^2$

$= 1$

2°, For each fixed $v_0 \in (-\infty, \infty)$,

$X(s, v_0) = (\cos s - v_0 \sin s, \sin s + v_0 \cos s, v_0)$

$= (\sqrt{1+v_0^2}\cos(s+\alpha_0), \sqrt{1+v_0^2}\sin(s+\alpha_0), v_0)$

where $\tan\alpha_0 = v_0$

$\therefore X$ maps S^1 to $\{(x,y,v_0) : x^2+y^2 = 1+v_0^2\}$ bijectively

$\therefore X$ is bijective from $S^1 \times (-\infty,\infty)$ to the hyperbolid.

3°, $\frac{\partial X}{\partial s} = (-\sin s - v\cos s, \cos s - v\sin s, 0)$

$\frac{\partial X}{\partial v} = (-\sin s, \cos s, 1)$

$\therefore X$ has rank 2 since $\{\frac{\partial X}{\partial s}, \frac{\partial X}{\partial v}\}$ is linearly independent.

② The catenoid S is given by $y^2 + z^2 = \cosh^2 x$

$\therefore$ We define $X: (-\infty,\infty) \times [0, 2\pi] \to S$ by

$X(s,t) = (s, \cosh s \cdot \cos t, \cosh s \cdot \sin t)$

$$\therefore \begin{cases} \frac{\partial X}{\partial s} = (1, \sinh s \cdot \cos t, \sinh s \cdot \sin t) \\ \frac{\partial X}{\partial t} = (0, -\cosh s \cdot \sin t, \cosh s \cdot \cos t) \end{cases}$$

$\therefore$ the first fundamental form is

$$[g_{ij}] = \begin{bmatrix} \cosh^2 s & 0 \\ 0 & \cosh^2 s \end{bmatrix}$$

③ $X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$

$\therefore \frac{\partial X}{\partial u} = (1-u^2+v^2,\ 2uv,\ 2u)$

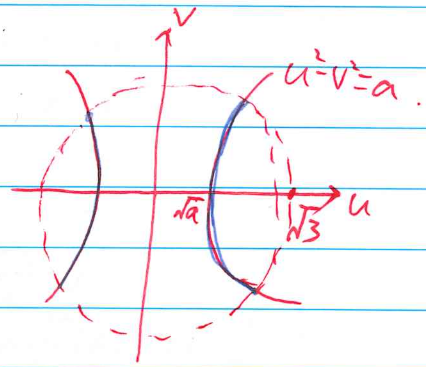
$\frac{\partial X}{\partial v} = (2uv,\ 1-v^2+u^2,\ -2v)$

1°. $\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = \left[-2u(1+u^2+v^2),\ 2v(1+u^2+v^2),\ 1-(u^2+v^2)^2\right]$

$\therefore \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = 0 \iff u = v = 0$ and $u^2+v^2 = 1$ contradiction

$\therefore X$ is regular

2°. When $u^2+v^2 < 3$



if $X(u_1, v_1) = X(u_2, v_2)$

then $u_1^2 - v_1^2 = u_2^2 - v_2^2 = a$ Assume $a \geq 0$

$\therefore (u_1, v_1), (u_2, v_2)$ lie on $u^2 - v^2 = a$

and $\begin{cases} u_1 - \frac{u_1^3}{3} + u_1 v_1^2 = u_2 - \frac{u_2^3}{3} + u_2 v_2^2 \\ v_1 - \frac{v_1^3}{3} + v_1 u_1^2 = v_2 - \frac{v_2^3}{3} + v_2 u_2^2 \end{cases}$

$\xrightarrow{u_i^2 - v_i^2 = a}$ $\begin{cases} (1-a)u_1 + \frac{2}{3}u_1^3 = (1-a)u_2 + \frac{2}{3}u_2^3 & \text{— ①} \\ (1+a)v_1 + \frac{2}{3}v_1^3 = (1+a)v_2 + \frac{2}{3}v_2^3 & \text{— ②} \end{cases}$

For ②, let $g(v) = (1+a)v + \frac{2}{3}v^3 \Rightarrow g'(v) = 1+a+2v^2 > 0$ since $a \geq 0$

$\therefore g$ is increasing

$\therefore g(v_1) = g(v_2) \Rightarrow v_1 = v_2$

$\because u_i^2 - v_i^2 = a$, $v_1 = v_2$

$\therefore u_1 = \pm u_2$

if $u_1 \neq u_2$, then $u_1 = -u_2$

$\therefore$ ① $\Rightarrow (1-a)u_1 + \frac{2}{3}u_1^3 = -\left[(1-a)u_1 + \frac{2}{3}u_1^3\right]$

$\therefore u_1\left[(1-a) + \frac{2}{3}u_1^2\right] = 0$

$\therefore u_1 = 0 \Rightarrow u_1 = u_2 = 0$, contradiction

$\therefore u_1 \neq 0$, $1 - a + \frac{2}{3}u_1^2 = 0$

$\therefore 0 = 1 - a + \frac{2}{3}u_1^2 \geq 1 - a + \frac{2}{3}a$ $[|u_1| \geq \sqrt{a}]$

$= 1 - \frac{a}{3} > 0$ since $a < 3$

$\therefore 0 > 0$, contradiction

$\therefore u_1 = u_2$ $\therefore X$ is injective for $u^2+v^2<3$ if $a \geq 0$

• For $a \le 0$, it is similar to $a \ge 0$

$\therefore$ X is injective on $u^2+v^2<3$

3°. $X(\sqrt{3},0)=(0,0,3)$

$X(-\sqrt{3},0)=(0,0,3)$

4°. $\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} = [-2u(1+u^2+v^2),\ 2v(1+u^2+v^2),\ 1-(u^2+v^2)^2]$

$\therefore$ $\left|\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}\right|^2 = (4u^2+4v^2)(1+u^2+v^2)^2 + [1-(u^2+v^2)^2]^2$

$= 4(u^2+v^2)[1+2(u^2+v^2)+(u^2+v^2)^2] + 1-2(u^2+v^2)^2+(u^2+v^2)^4$

$= 4(u^2+v^2)+8(u^2+v^2)^2+4(u^2+v^2)^3+1-2(u^2+v^2)^2+(u^2+v^2)^4$

$= 1+4(u^2+v^2)+6(u^2+v^2)^2+4(u^2+v^2)^3+(u^2+v^2)^4$

$= [1+(u^2+v^2)]^4$

$\therefore$ $\vec{N} = \frac{\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}}{\left|\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}\right|} = \left[\frac{-2u}{1+u^2+v^2},\ \frac{2v}{1+u^2+v^2},\ \frac{1-(u^2+v^2)^2}{[1+u^2+v^2]^2}\right]$

$= \left[\frac{-2u}{1+u^2+v^2},\ \frac{2v}{1+u^2+v^2},\ \frac{[1-(u^2+v^2)][1+(u^2+v^2)]}{[1+(u^2+v^2)]^2}\right]$

$= \frac{1}{1+u^2+v^2}[-2u,\ 2v,\ 1-(u^2+v^2)]$

④ Let S be a surface in $\mathbb{R}^3$, and $P \in \text{int}(S)$.
Let X, Y be two parametrization of S around P
Then $\exists\ \mathcal{U}, \mathcal{U}'$ two open subsets of $\mathbb{R}^2$ such that

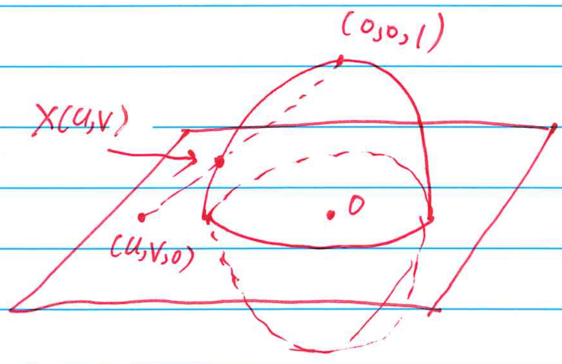
$X: \mathcal{U} \to S$
$Y: \mathcal{U}' \to S$ are both bijective.

Let $\delta = Y^{-1} \circ X : \mathcal{U} \to \mathcal{U}'$
then δ is smooth and bijective near $X^{-1}(P)$.
and we have $X = Y \circ \delta$

$$\therefore \begin{cases} \partial_u X = \partial_{u'} Y \cdot \frac{\partial u'}{\partial u} + \partial_{v'} Y \cdot \frac{\partial v'}{\partial u} \in \text{span}\{\partial_{u'} Y, \partial_{v'} Y\} \\ \partial_v X = \partial_{u'} Y \cdot \frac{\partial u'}{\partial v} + \partial_{v'} Y \cdot \frac{\partial v'}{\partial v} \in \text{span}\{\partial_{u'} Y, \partial_{v'} Y\} \end{cases}$$ at the point P.

$\therefore$ $\text{span}\{\partial_u X, \partial_v X\} \subseteq \text{span}\{\partial_{u'} Y, \partial_{v'} Y\}$
$\because$ δ is smooth and bijective near $X^{-1}(P)$.
$\therefore$ Similarly, $\text{span}\{\partial_{u'} Y, \partial_{v'} Y\} \subseteq \text{span}\{\partial_u X, \partial_v X\}$.
$\therefore$ the tangent spaces are the same.

⑤ 1°, From the graph,



$X(u,v) = (0,0,1) + t_0(u,v,-1)$ for some t_0.
$\because$ $X(u,v)$ lies on the unit sphere
$\therefore$ $|X(u,v)|^2 = 1$
$t_0^2 u^2 + t_0^2 v^2 + (1-t_0)^2 = 1$
$(1+u^2+v^2)t_0^2 - 2t_0 = 0$
$\therefore$ $t_0 = 0$ or $t_0 = \dfrac{2}{1+u^2+v^2}$

when $t_0 = 0$, $X(u,v) = (0,0,1)$, this is not what we want.
$\therefore$ We define $X: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$X(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right)$$

2°, $\because$ the z-coordinate of $X(u,v)$ is $\dfrac{-1+(u^2+v^2)}{1+(u^2+v^2)}$.
$\therefore$ when $u^2+v^2<1$, $X(u,v)$ lies on the southern hemisphere.
when $u^2+v^2>1$, $X(u,v)$ lies on the northern hemisphere.

$3°$, $\frac{\partial x}{\partial u} = \left(\frac{2(1-u^2+v^2)}{[1+u^2+v^2]^2}, \frac{-4uv}{[1+u^2+v^2]^2}, \frac{4u}{[1+u^2+v^2]^2} \right)$

$\frac{\partial x}{\partial v} = \left(\frac{-4uv}{[1+u^2+v^2]^2}, \frac{2(1+u^2-v^2)}{[1+u^2+v^2]^2}, \frac{4v}{[1+u^2+v^2]^2} \right)$

$\therefore$ the first fundamental form is

$$g_{ij} = \begin{bmatrix} \frac{4}{[1+u^2+v^2]^2} & 0 \\ 0 & \frac{4}{[1+u^2+v^2]^2} \end{bmatrix}$$

(6) $1°$, $\alpha(t) = (\sin t \cos u_0, \sin t \cdot \sin u_0, \cos t)$, $0 < a \le t < b < \pi$.

$\therefore$ the length of $\alpha(t)$ is

$$L(\alpha) = \int_a^b |\alpha'(t)|\, dt$$

$$= \int_a^b 1\, dt = b-a.$$

$2°$, $\because \beta(t) = X(u(t), v(t))$, $a \le t \le b$

$\therefore \beta'(t) = u'(t) \cdot X_u + v'(t) \cdot X_v$

$\therefore \int_a^b \langle \beta'(t), \partial_v X \rangle \le \int_a^b |\beta'(t)| \cdot |\partial_v X|\, dt = \int_a^b |\beta'(t)|\, dt = L(\beta)$ since $|\partial_v X| = 1$.

$= \int_a^b u'(t) \langle \partial_u X, \partial_v X \rangle + v'(t) \langle \partial_v X, \partial_v X \rangle.$

$= \int_a^b v'(t)\, dt$

$= v(b) - v(a)$

$= b - a$ since $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$

$\therefore L(\alpha) \le L(\beta)$.

[we take $\langle \beta', \partial_v X \rangle$ is because $\partial_v X$ is "pointing" from $\alpha(a)$ to $\alpha(b)$].

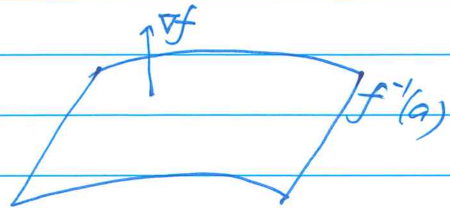
⑦ 1°: $\because$ a is a regular value of f and $M = f^{-1}(a)$

$\therefore$ $\nabla f \neq 0$ on M.

$\therefore$ We define

$$N = \frac{\nabla f}{|\nabla f|} = \frac{(f_x, f_y, f_z)}{|(f_x, f_y, f_z)|}$$

$\therefore$ N is well-define and a unit vector.



2°: Let $X: V \subseteq \mathbb{R}^2 \to M$ be any paramotenization of M.

$\therefore$ $f(X(u,v)) = 0$ for any $(u,v) \in V$

$\therefore$ $\partial_u[f(X(u,v))] = \nabla f \cdot X_u = 0$

$\partial_v[f(X(u,v))] = \nabla f \cdot X_v = 0$

$\therefore$ $\nabla f \perp TM$

$\therefore$ N is a unit normal vector field of M.